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The stability of a gravitating fluid layer of infinite extent but finite thickness with a force-free magnetic field

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The stability of a gravitating, homogeneous, incompressible and infinitely conducting fluid layer of infinite extent but finite thickness is investigated in presence of a force-free magnetic field. The critical wave-numbers for the onset of instability are calculated for some assigned strengths of magnetic field. It is found that stability increases with the increase of mean magnetic field. The wave-length for maximum instability and corresponding maximum growth rate of instability are calculated.

1. INTRODUCTION

Safranov (1960) pointed out the importance of the study of gravitational and magneto-gravitational stability of fluid layers in astronomical context. Oganesyan (1961) studied the gravitational and magneto-gravitational stability of a fluid layer of infinite extent but finite thickness. The effect of uniform rotation on the stability of a fluid layer was studied by Chakraborty (1964) and in more details by Uberoi (1963). The effects of Hall current and finite resistance on the stability of such a layer were studied respectively by Das (1968) and Sundaran (1968). Here we study the stability of a gravitating, homogeneous, incompressible and infinitely conducting fluid layer of infinite extent but finite thickness in presence of a force-free magnetic field. The effect of a force-free magnetic field on the stability of a infinitely long gravitating cylinder was discussed by Trehan (1958). The energy method of Chandrasekhar & Fermi (1953) is applied here for examining stability. A table is given for the critical wave-number for the onset of instability for some assigned value of the mean magnetic field. The wave-length for maximum instability and the corresponding maximum growth rate of instability are calculated.

2. THE FORCE-FREE MAGNETIC FIELD IN EQUILIBRIUM CONFIGURATION

Let a uniform, homogeneous, gravitating plasma occupy the space bounded by the planes $z = h$ and $z = -h$. The remaining space is a vacuum.

Force-free magnetic field $\vec{H}$ satisfies the equations

$$\text{rot } \vec{H} = \alpha \vec{H} \qquad \ldots(1)$$

$$\text{and div } \vec{H} = 0 \qquad \ldots(2)$$

in which α is a constant.

The magnetic field $\vec{H}$ can be obtained from a scalar function ψ, by the relation (Pulpton & Ferraro 1961)

$$\vec{H} = \text{rot}\ (\vec{a}\psi) + \frac{1}{\alpha}\ \text{rot}\ \ (\vec{a}\psi) \qquad ...(3)$$

and ψ satisfied the equation

$$\nabla^2\psi + \alpha^2\psi = 0 \qquad ...(4)$$

Let us suppose that the field is independent of y coordinate and its dependence on x coordinate is periodic. Then the solution of equation 4 is

$$\psi = A \cos \gamma z \cos \beta x \qquad ...(5)$$

in which β is a constant and

$$\gamma^2 = \alpha^2 - \beta^2 \qquad ...(6)$$

Therefore by equation (3), taking $\vec{a} = \hat{x}$, a unit vector along x-axis, we have

$$\vec{H} = \hat{x}\ \frac{A\gamma^2}{\alpha}\ \cos\ \gamma z\ \cos \beta x + \hat{y}\ A\ \gamma\ \sin \gamma z\ \cos \beta x + \hat{z}\ \frac{\beta\gamma A}{\alpha}\ .$$
$$\sin \gamma z \sin \beta x \quad ...(7)$$

Let at the boundary $z = \pm h$, and the magnetic field be directed along x-axis and vanish outside. Therefore we must have

$$\sin \gamma h = 0$$

or, $$\gamma h = n\pi \qquad ...(8)$$

in which n is any integer.

Averaging $|\vec{H}|^2$ over x we get

$$< |\vec{H}|^2 > = \frac{1}{2} \cdot \frac{A^2\gamma^2}{\alpha^2}\ [(\beta^2 + \alpha^2) \sin^2 \gamma z + \gamma^2 \cos^2 \gamma z] \qquad ...(9)$$

The average of the square of the magnetic field per unit length along x-axis is given by,

$$\ll |\vec{H}|^2 \gg = \frac{1}{2}\ \frac{A^2\gamma^2}{\alpha^2 h}\int_0^h [(\beta^2 + \alpha^2) \sin^2 \gamma z + \gamma^2 \cos^2 \gamma z]dz$$

$$= \frac{1}{2} A^2\gamma^2 \qquad ...(1)$$

Let H_m^2 denote the average of the square of the magnetic field, then

$$H_m^2 = \ll |\vec{H}^2| \gg = \frac{1}{2} A^2\gamma^2 = \frac{A^2 n^2 \pi^2}{2h^2} \qquad ...(11)$$

3. The examination of stability

Following Chandrasekhar & Fermi (1953), let the layer be given a small perturbation in such a way that its boundary is given by

$$z = \pm (h + a \cos kx) \qquad \text{...(12)}$$

in which $\frac{a}{h} \ll 1$ and $\frac{2\pi}{k}$ is the wave-length of disturbance in x-direction.

The displacement $\vec{\xi}$ at any point is given by

$$\vec{\xi} = \text{grad}\ \phi \qquad \text{.. (13)}$$

in which ϕ satisfies the equation

$$\nabla^2\phi = 0 \qquad \text{.. (14)}$$

The solution of (14) that gives displacements symmetrical about the plane $z=0$, is given by

$$\phi = B \cosh kz \cos kx \qquad \text{.. (15)}$$

The displacements are given by

$$\xi_x = - Bk \cosh kz \sin kx$$
$$\xi_z = Bk \sinh kz \cos kx$$
$$\xi_y = 0 \qquad \text{...(16)}$$

Since at $z = h$, ξ_z must reduce to $a \cos kx$

$$B = \frac{a}{k \sinh kh} \qquad \text{...(17)}$$

Now we find the change in potential energy and the change in magnetic energy due to the perturbation.

(a) *Change in potential energy*

Let U and V be the internal and external potentials respectively. These two potentials satisfy equations

$$\triangle^2 U = 0 \quad \text{and} \quad \triangle^2 V = - 4\pi G\rho \qquad (18)$$

in which G is gravitational constant and ρ is the density of the medium. The solution of equations (18) to the first order in a are

$$U = - 4\pi G\rho hz + aAe^{-kz} \cos kz, \quad z>h$$
$$V = - 2\pi G\rho z^2 + aB \cosh kz \cos kx \qquad \text{...(19)}$$

The constants A, B are obtained from the conditions that U and V and its derivatives with respect to z are continuous across $z = h + a \cos kx$. These two conditions give

$$Ae^{-kh} = B\cosh kh$$
$$Ae^{-kh} + B\sinh kh = \frac{4\pi G\rho}{k}.$$

Solving we get $A = \frac{4\pi G\rho}{k}\cos kh$, $B = \frac{4\pi G\rho}{k}e^{-kh}$ ...(20)

The change in gravitational potential energy per unit length along x and y axes due to infinitesimal displacements

$$\delta\xi_x = -\frac{\delta a}{\sinh kh}\cosh kz \sin kz \quad \text{and} \quad \delta\xi_z = \frac{\delta a}{\sinh kh}\sinh kz\cos kx$$

is given by,

$$\delta(\triangle\Omega) = -2\rho\left[\int_0^{h+a\cos kx}(\delta\xi.\mathrm{grad}\,V)dz\right]_{\text{averaged over } x}$$

$$= -2\rho\left[\int_0^{h+a\cos kx}\left\{\frac{\delta a}{\sinh kh}\sinh kz\cos kx\,(-4\pi G\rho z + 4\pi G\rho a e^{-kh}\sinh kz\cos kx)\right.\right.$$

$$\left.\left. + \frac{\delta a}{\sinh kh}\cosh kz\sin kx(4\pi G\rho a e^{-kh}\cosh kz\sin kx)\right\}dx\right]_{\text{averaged over } x}$$

$$= 4\pi G\rho^2 h\,a\delta a\left[1 - \frac{1}{2kh}\left(1 + e^{-2kh}\right)\right] \quad ...(21)$$

Integrating this between the limits 0 to a, we get the change in gravitational potential energy due to displacements given by (16)

$$\triangle\Omega = H_s^2\left[1 - \frac{1}{2kh}(1 + e^{-2kh})\right] \quad ...(22)$$

in which $H^2_s = 2\pi G\rho^2 h^2$

(b) *Change in magnetic energy*

By equation (33) of Trehan (1958) we have for the change in magnetic energy per unit length along x and y axes,

$$\triangle M = \frac{1}{8\pi}\left[2\int_0^h\left|\left(\vec{H}.\,\mathrm{grad}\,\vec{\xi}\right)\right|^2 dz\right]_{\text{averaged over } x}$$

$$= \frac{1}{4\pi}\left[\int_0^h\left\{\frac{\beta^2\gamma^2A^2a^2k^2}{\alpha^2\sinh^2 kh}\left(\sin^2\gamma z\cosh^2 kz\sin^2\beta x\cos^2 kx + \sin^2\gamma z\sinh^2 kz\right.\right.\right.$$

$$\left.\sin^2\beta x\sin^2 kx\right) + \frac{\gamma^2A^2a^2k^2}{\alpha^4\sinh^2 kh}\left(\cos^2\gamma z\sinh^2 kz\cos^2\beta x\sin^2 kx + \cos^2\gamma z\right.$$

$$\left.\left.\left.\cosh^2 kz\cos^2\beta x\cos^2 kx\right)\right\}dz\right]_{\text{av.}x} = \frac{(Hm^2/8\pi)a^2k\epsilon}{8^2\sinh kh}\;\frac{\beta^2h^2 - n^2\pi^2}{\beta^2h^2 + n^2\pi^2}$$

$$\left[(1+\delta)\frac{n^2\pi^2(2k^2h^2 + \beta^2h^2 + n^2\pi^2)}{(k^2h^2 + n^2\pi^2)(\beta^2h^2 - n^2\pi^2)}\sinh 2kh + 2(1-\delta)kh\right] \quad ..(23)$$

in which the following relations have been used

$$[\sin^2 \beta x \cos^2 kx]_{av.x} = \frac{1}{4}\epsilon, \quad [\cos^2 \beta x \sin^2 kx]_{av.x} = \frac{1}{4}\epsilon$$

$$[\sin^2 \beta x \sin^2 kx]_{av.x} = \frac{1}{4}\epsilon\,\delta, \quad [\cos^2 \beta x \cos^2 kx]_{av.x} = \frac{1}{4}\epsilon\delta$$

where $\epsilon = 1,\ \delta = 1,$ if $\beta \neq k$

and $\epsilon = \frac{1}{2},\ \delta = 3,$ if $\beta = k$ ...(24)

Thus the total change in energy is given by

$$E = \triangle\Omega + \triangle M = \frac{H_s'^2 a^2}{h} f(kh) \qquad ...(25)$$

in which $f(kh)$ is given by

$$f(kh) = 1 - \frac{1}{2kh}(1 + e^{-2kh}) + \frac{H_m^2}{8\pi H_s^2} \cdot \frac{kh\epsilon}{8\sinh^2 kh} \cdot \frac{\beta^2 h^2 - n^2\pi^2}{\beta^2 h^2 + n^2\pi^2}$$

$$\left[(1+\delta)\frac{n^2\pi^2(2k^2h^2 + \beta^2h^2 + n^2\pi^2)}{(k^2h^2 + n^2\pi^2)(\beta^2h^2 - n^2\pi^2)}\sinh 2kh + 2(1-\delta)kh\right] \qquad ...(26)$$

To find the Lagrangian function, we shall have to find the kinetic energy of the motion resulting from the varying amplitude. Since the fluid is incompressible the velocity components can be obtained from a velocity potential ψ, through the relation.

$$\vec{u} = \text{grad}\,\psi \qquad ...(27)$$

and ψ satisfies the Laplace's equation. The solution for ψ that gives velocity components symmetric about the plane $z = 0$ is,

$$\psi = A\cosh kz \cos kx \qquad ..(28)$$

Therefore, $u_x = \dfrac{\partial\psi}{\partial x} = -Ak\cosh kz \sin kx$

$$u_z = \frac{\partial\psi}{\partial x} = Ak\sinh kz \cos kx \qquad ...(29)$$

At $z = h$ we must have

$$\frac{da}{dt}\cos kx = uz/_{z=h} = Ak\sinh kh \cos kx$$

$$\text{or,} \quad A = \frac{1}{k\sinh kh}\frac{da}{dt} \qquad ..(30)$$

The kinetic energy per unit length along x and y axes is thus given by

$$T = \frac{1}{2}\,\rho A^2 k^2 \left[2 \int_0^h \left(\sinh^2 kz \cos^2 kx + \cosh^2 kz \sin^2 kx \right) dz \right]_{\text{averaged over } x}$$

$$= \frac{\rho}{2k} \coth kh \left(\frac{da}{dt} \right)^2 \qquad \ldots(31)$$

Therefore the Lagrangian function L is

$$L = T - E = \frac{\rho}{2k} \coth kh \left(\frac{da}{dt} \right)^2 - \frac{H_s^2 a^2}{h} f(kh) \qquad \ldots(32)$$

The equation of motion derived from this Lagrangian is

$$\frac{d^2 a}{dt^2} = -\frac{4\pi G \rho kh}{\coth kh}\, a f(kh) \qquad \ldots(33)$$

Let the solution for a be

$$a = \text{constant} \,.\, e^{\pm qt} \qquad \ldots(34)$$

then

$$q^2 = -\frac{4\pi G \rho\, kh}{\coth kh} f(kh) \qquad \ldots(35)$$

As $\coth kh > 0$, this small motion is stable if $f(kh) > 0$ and unstable if $f(kh) < 0$. As there is a single positive root X^* of the equation $f(X) = 0$ and $f(X) < 0$ when $X < X^*$; $f(X) > 0$ when $X > X^*$, all the modes of deformation with $X < X^*$, are unstable. Here $X = kh$. Thus all modes of deformation whose wave-lengths are greater than $2\pi h / X^*$ are unstable.

If $\theta = \frac{H_m^2}{8\pi H_s^2} > 1$, $X \ll 1$, in this case q^2 is given by (keeping terms up to X^2),

$$q^2 = 4\pi G \rho \left[X - 2X^2 - \theta X^2 \left\{ \frac{\epsilon(\beta^2 h^2 - n^2\pi^2)}{4(\beta^2 h^2 + n^2 \pi^2)} (1 - \delta) + \frac{1}{4}\epsilon(1 + \delta) \right\} \right] \qquad \ldots(36)$$

Let q^2 be maximum when $X = X_m = \frac{2\pi h}{\lambda_m}$ and let the corresponding value of q be q_m.

From equation (36), we get the following expressions for λ_m and q_m.

$$\lambda_m = 4\pi h\left[2 + \theta\left\{\frac{\epsilon(\beta^2h^2 - n^2\pi^2)}{4(\beta^2h^2 + n^2\pi^2)}(1-\delta) + \frac{1}{4}\epsilon(1+\delta)\right\}\right] \quad \ldots(37)$$

$$q_m = 2\pi G\rho X_m = \sqrt{\pi G\rho}\left[2 + \theta\left\{\frac{\epsilon(\beta^2h^2 - n^2\pi^2)}{4(\beta^2h^2 + n^2\pi^2)}(1-\delta) + \frac{1}{4}\epsilon\delta(1+\delta)\right\}\right]^{-1/2} \quad . (38)$$

This λ_m is the wave-length of disturbance, that gives maximum instability and q_m is the corresponding maximum growth rate of instability.

For non-resonance case i.e., $\beta \neq k$, $f(X)$ is given by

$$f(X) = 1 - \frac{1}{2X}(1+e^{-2X}) + \theta.\frac{n^2\pi^2}{\beta^2h^2+n^2\pi^2}\cdot\frac{X(2X^2+\beta^2h^2+n^2\pi^2)}{2(X^2+n^2\pi^2)}\cdot \coth X \quad \ldots(39)$$

The roots X^* of the equation $f(X) = 0$, where $f(X)$ is given by the above eqation (39) with $n = 1$, are given in the following table for some values of θ and βh

From this table it is obserbed that stability increases with the increase of θ and also with the decrease of βh.

β^2h^2 \ θ	0·25	0·50	0·75	1·00
0·50	0·5759	0·5270	0·4874	0·4544
0·75	0·5760	0·5271	0·4875	0·4545
1·00	0·5761	0·5272	0·4877	0·4546
2·00	0·5764	0·5275	0·4881	0·4550

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References

Chakraborty, B.B. 1964 *Indian. J. Phys.* **38**, 490.

Chandrasekhar, S & Fermi, F. 1953 *Astrophys. J.* **118**, 116.

Das, K. P. 1968 *Canadian. J. Phys.* **46**, 2201

Oganesyan, R.S. 1961 *Soviet Astronomy*, **4**, 434, 634.

Pulpton, C & Ferraro, V.C.A. 1961 *An Introduction to Magneto-Fluid Mechanics*; Oxford University Press P. 36.

Safranov, V.S. 1960 *Ann. Astrophys* **23**, 979.

Sundaran, A.K. 1968. *Phys. Fluids* **11**, 1709.

Trehan S.K. 1958 *Astrophys. J.* **127**, 436.

Uberoi, C. 1963 *J. Indian. Inst. Sci.* **46**, 11.